The Likelihood Ratio Test of Common Factors under Non-Ideal Conditions

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ABSTRACT: The Spatial Durbin model occupies an interesting position in Spatial Econometrics. It is the reduced form of a model with cross-sectional dependence in the errors and it may be used as the nesting equation in a more general approach of model selection. Specifically, in this equation we can obtain the Likelihood Ratio test of Common Factors (LR\text{COM}). This test has good properties if the model is correctly specified, as shown in Mur and Angulo (2006). However, as far as we know, there is no literature in relation to the behaviour of the test under non-ideal conditions, which is the purpose of the paper. Specifically, we study the performance of the test in the case of heteroscedasticity, non-normality, endogeneity, dense weighting matrices and non-linearity. Our results offer a positive view of the Likelihood Ratio test of Common Factors, which appears to be a useful technique in the toolbox of spatial econometrics.

JEL Classification: C21, C50, R15.

Keywords: Likelihood Ratio Test of Common Factor, Heteroscedasticity, Non-normality, Endogeneity, Non-linearity.

El Ratio de Verosimilitudes de Factores Comunes bajo condiciones no ideales

RESUMEN: El modelo espacial de Durbin ocupa una posición interesante en econometría espacial. Es la forma reducida de un modelo de corte transversal con dependencia en los errores y puede ser utilizado como ecuación de anidación en un enfoque más general de selección de modelos. En concreto, a partir de esta ecuación puede obtenerse el Ratio de Verosimilitudes conocido como test de Factores Comunes (LR\text{COM}). Como se muestra en Mur y Angulo (2006), este test tiene bue-
nas propiedades si el modelo está correctamente especificado. Sin embargo, por lo que sabemos, no hay referencias en la literatura sobre el comportamiento de este test bajo condiciones no ideales. En concreto, estudiamos el comportamiento del test en los casos de heterocedasticidad, no normalidad, endogeneidad, matrices de contactos densas y no-linealidad. Nuestros resultados ofrecen una visión positiva del test de Factores Comunes que parece una técnica útil en el instrumental propio de la econometría espacial contemporánea.

Clasificación JEL: C21, C50, R15.

Palabras clave: Contraste de Ratio de Verosimilitudes de Factores Comunes, Heterocedasticidad, No Normalidad; Endogeneidad, No Linealidad.

1. Introduction

In recent years, there has been an increasing concern about questions related to methodology in Spatial Econometrics. The works of Anselin and Florax (1995), Anselin et al (1996) and Anselin and Bera (1998) played a leading role in the revitalisation of the interest in the nineties. These papers underline the difficulties arising from the lack of specificity of the tests based on the Lagrange Multiplier principle and, consequently, the problems of finding the true model when there are various alternatives. In sum, there is a serious risk of obtaining a misspecified model if the user is not sufficiently careful.

In a model selection context, two main strategies can be identified. The first starts with a general model that we try to simplify in a so-called «General-to-Specific» approach (Hendry, 1980). This strategy has been supported by an important part of the literature on econometric model selection (Danilov and Magnus, 2004; Hendry and Krolzig, 2005). The second approach, denoted as «from Specific-to-General», operates in the opposite direction: starts from a simple model that it is extended depending on the results for certain tests. Comparison of both strategies have been numerous (Campos et al, 2005; Lütkepohl, 2007), also in the context of spatial econometrics. Florax et al (2003, 2006) compared the two approaches under ideal conditions while Mur and Angulo (2009) introduce different anomalies in the Data Generating Process (DGP). Elhorst (2010) reviews the situation once again.

As indicated in Florax et al (2006) or Mur and Angulo (2009) the starting point of the General-to-Specific strategy is the Spatial Durbin Model (SDM form now on) or, in other words, an «autoregressive distributed lag model of the first order» as defined by Bivand (1984). Lesage and Pace (2009, p. 46) are in favour of the SDM which «provides a general starting point for discussion of spatial regression model estimation since this model subsumes the spatial error model and the spatial autoregressive model». Elhorst (2010) remarks some of the strengths of the SDM: i) «it produces unbiased coefficient estimates also if the true data-generation process is a spatial lag or a spatial error model»; ii) «it does not impose prior restrictions on the magnitude of potential spatial spillover effects», which can be global or local and/or different
for different explanatory variables; and iii) «it produces correct standard errors or t-values of the coefficient estimates also if the true data-generating process is a spatial error model». In addition, Elhorst (2010) proposes a test procedure to select the most adequate model which confers an important role to the SDM.

The Spatial Lag Model (SLM) is a particular case of the SDM, when the exogenous interaction effects among the independent variables are not significant. The Spatial Error Model (SEM) is also a particular case of the SDM, once the common factor hypothesis in introduced in the SDM model. Hence, if the null is not rejected the test favours the SEM specification. When the null is not rejected, Florax et al (2006) propose to select the Spatial Lag Model (SLM) while Mur and Angulo (2009) and Elhorst (2010) propose to go on testing further hypotheses on the SDM. It is clear that the last equation plays a crucial role in the specification of a spatial model. For this reason it is important to be aware of the weaknesses and strengths of the specification tests applied, like the Likelihood Ratio test of Common Factors (LR_{COM} in what follows), on this equation.

However, the literature on Spatial Econometrics has paid little attention to the Common Factor test. This is a bit surprising. To cite only some of the most recent cases, this test is not included in the comprehensive simulation carried out by Anselin and Florax (1995), nor is it mentioned in the meta-analysis of Florax and de Graaff (2004); the LR_{COM} test does not appears in the manuals of Tiefelsdorf (2000) and Griffith (2003). On the contrary, Lesage and Pace (2009) are very confident about the possibilities of the test. Recently, Mur and Angulo (2006) conducted a Monte Carlo exercise in order to evaluate the behaviour of the test under ideal conditions. In this paper, we go further in the same direction by analysing the performance of the Likelihood Ratio test of Common Factors$^1$ under non-ideal conditions: heteroscedasticity, non-normality, non-linearity, endogeneity and dense weighting matrices.

The paper is organised as follows. The next section describes the Spatial Durbin model and the Likelihood Ratio test of Common Factors following Mur and Angulo (2009) for the definition of the alternative hypothesis. Section 3 describes a Monte Carlo experiment that provides evidence on the performance of the test for various departures from the case of ideal conditions. The main conclusions are summarised in Section 4.

2. The Spatial Durbin Model and the Likelihood Ratio test of Common Factors

The Durbin model plays a major role in a General-to-specific strategy of model selection. Following a Hendry-like approach, it is a general equation that nests two of

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$^1$ We focus on the Likelihood Ratio version of the test of Common Factors because, in general, it is better-known. Two other alternatives are the Wald and the Lagrange Multiplier versions, as developed by Burridge (1981).
the most popular models in spatial econometrics, the Spatial Lag Model (SLM) and
the Spatial Error Model (SEM). Let’s analyse this issue more in detail.

The Durbin Model appears in a specific situation in which, using time series, we
need to estimate an econometric model with an autoregressive error term, AR(1):

\[
\begin{align*}
    y_t &= x_t' \beta + u_t \\
    u_t &= \rho u_{t-1} + \varepsilon_t
\end{align*}
\]

Durbin (1960) suggested directly estimating the reduced unrestricted form of (1)
by least squares:

\[
y_t = \rho y_{t-1} + x_t' \beta + x_{t-1} \eta + \varepsilon_t
\]

The adaptation of these results to the spatial case does not involve any special
difficulty, as shown by Anselin (1980):

\[
y = x' \beta + u \\
\Rightarrow y = \rho Wy + x' \beta + Wx \eta + \varepsilon
\]

where \( W \) is the weighting matrix; \( y, u \) and \( \varepsilon \) are vectors of order \((R \times 1)\); \( x \) is the \((R \times k)\)
matrix of observations of the \( k \) regressors; \( \beta \) and \( \eta \) are \((k \times 1)\) vectors of parameters
and \( \rho \) is the parameter of the spatial autoregressive process of the first order, SAR(1),
that intervenes in the equation of the errors.

We complete the specification of the model of (3) with the additional assumption
of normality in the random terms:

\[
y = \rho Wy + x' \beta + Wx \eta + \varepsilon
\]

This model can be estimated by maximum-likelihood (ML in what follows). The
log-likelihood function is standard:

\[
l(y / \varphi) = -\frac{R}{2} \ln 2\pi - \frac{R}{2} \ln \sigma^2 - \frac{1}{2} \ln \det B + \ln |B|
\]

with \( \varphi = [\beta, \eta, \rho, \sigma^2] \); \( B \) is the matrix \([I - \rho W]\) and \( |B| \) its determinant, the Jacobian
term.

Starting from the Durbin model, we can test whether or not some simplified
models such as the SLM, SEM or a purely static model without spatial effects are
admissible. Figure 1 summarizes the relationship between the four models.
Starting from the general SDM model, if we cannot reject the null hypothesis that the spatial lag of the x variable is not significant, $H_0: \eta = 0$, the evidence points to an SLM model or to a static model, depending on what happens with the parameter $\rho$. Hence, the next step consists on the estimation of the SLM model:

$$y = \rho Wy + x\beta + \epsilon$$

with $\epsilon \sim N(0, \sigma^2 I)$. 

Finally, the null hypothesis that $\rho = 0$ needs to be tested. If this assumption cannot be maintained, the evidence is in favour of the SLM model; otherwise, a simple static model should be the final specification:

$$y = x\beta + \epsilon$$

with $\epsilon \sim N(0, \sigma^2 I)$. 

In relation to the SEM model, the Common Factor hypothesis should be tested directly in the SDM equation, which results in $k$ non-linear restrictions: $\eta = -\rho \beta$ on the parameters of the equation. The most popular test in this context is the Likelihood Ratio of Common Factors, $L_{\text{RCOM}}$, proposed by Burridge (1981).

Introducing the $k$ non-linear restrictions on the model of (4), we obtain a SEM specification:

$$y = x\beta + u$$

$$u = \rho Wu + \epsilon$$

whose log-likelihood function is also standard:

$$l(y / \phi_0) = -\frac{R}{2} \ln 2\pi - \frac{R}{2} \ln \sigma^2 - \frac{\left(y - x\beta\right)' B' B (y - x\beta)}{2\sigma^2} + \ln |B|$$

with $\phi_0 = [\beta, \rho, \sigma^2]'$. 

**Figure 1.** Relationships between different spatial models for cross-sectional data
The log Likelihood Ratio compares the maximized values of the log-likelihoods of the models (5) and (9):

\[ H_0 : \rho \beta + \eta = 0 \Rightarrow LR_{COM} = 2 \left[ l(y / \tilde{\phi}_A) - l(y / \tilde{\phi}_0) \right] \sim \chi^2 (k) \]  

As in the previous case, if we cannot reject the null hypothesis, the evidence points to a SEM model or to a static model, depending on the significance test of \( \rho \).

Let us finish this section highlighting the most important points, according to our own perspective:

(i) The Spatial Durbin Model occupies a prominent role in the specification process of a spatial model, because it nests other simpler models.

(ii) The connection between the SDM and the SLM model is a single significance test of a maximum-likelihood estimate, whose properties are very well-known.

(iii) The connection between the SDM and the SEM is the Common Factor Test. The Likelihood Ratio version, LR_{COM}, is simple to obtain but its properties are known only under ideal conditions.

### 3. The LR_{COM} test under non-ideal conditions. A Monte Carlo analysis.

In this section, we evaluate the performance of the LR_{COM} test in different non-ideal situations and for different sample sizes. Section 3.1 describes the characteristics of the experiments and Section 3.2 focuses on the results.

#### 3.1. Design of the Monte Carlo

We use a simple linear model as a starting point:

\[ y = x \beta + \varepsilon \]  

where \( x \) is an \((R \times 2)\) matrix whose first column, made of ones, is associated to the intercept whereas the second corresponds to the regressor, \( x_r \); \( \beta \) is a \((2 \times 1)\) vector of parameters, \( \beta' = [\beta_0; \beta_1] \), and \( \varepsilon \) is the \((R \times 1)\) vector of error terms. From this expression, it is straightforward to obtain a Spatial Error Model, SEM, or a Spatial Lag Model, SLM. In matrix terms:

\[
\begin{align*}
\text{SEM:} \\
y &= x \beta + u \\
u &= \rho W u + \varepsilon \\
\varepsilon &\sim iid(0; \sigma^2 I)
\end{align*}
\]
SLM: \[
\begin{align*}
    y &= \rho Wy + x\beta + \varepsilon \\
    \varepsilon &\sim iid(0; \sigma^2 I)
\end{align*}
\]

The SEM and the SLM specification of (12a) and (12b) are the two alternative DGPs that we introduce in our simulation (other alternatives are also possible; Elhorst, 2010). The main characteristics of the exercise are the following:

a) Only one regressor has been used in the model. The coefficient associated takes a value of 2, $\beta_1 = 2$, whereas the intercept is equal to 10, $\beta_0 = 10$. Both magnitudes guarantee that, in the absence of spatial effects, the expected $R^2$ is 0.8.

b) The observations of the $x$ variable and of the random terms $\varepsilon$ and $u$ have been obtained from a univariate normal distribution with zero mean and unit variance. That is, $\sigma^2$ is equal to one in all the cases.

c) We have used three different sample sizes, $R$, with 49, 100 and 225 observations distributed in regular grids of $(7 \times 7)$, $(10 \times 10)$ or $(15 \times 15)$, respectively. The weighting matrix is the row-normalized version of the original rook-type binary matrix.

d) In each case, 11 values of the parameter $\rho$ have been simulated, only on the non-negative range of values, $\{\rho = 0; 0.1; 0.2; 0.3; 0.4; 0.5; 0.6; 0.7; 0.8; 0.9; 0.95\}$.

e) Each combination has been repeated 1000 times.

The two DGPs, SEM or SLM, have been simulated under different conditions, as follows:

i. Ideal conditions. This is the control case that corresponds to expressions (12a) and (12b), in which all the hypotheses are met.

ii. Heteroscedasticity. The error terms are obtained from a normal distribution with non-constant variance: $\varepsilon_r \sim N(0; \alpha^2 \text{het}_r)$, where $\text{het}_r$ reflects the corresponding mechanisms of heteroscedasticity. In this case, we have used two spatial heteroscedasticity patterns, denoted as $h_1$ and $h_2$, and a non-spatial pattern, $h_3$. The skedastic function for the first two cases is: $\text{het}_r = d(a, r)$ being $d(\cdot)$ a normalized measure of distance between the centroids of the cells $a$ and $r$. In the $h_1$ case, $a$ is the cell situated in the upper-left corner of the lattice, whereas, in $h_2$, this cell is located in the centre of the lattice. The skedastic function in the case $h_3$ is $\text{het}_r = |x_r|$, a non-spatial pattern that depends on the realization of the regressor, $x_r$, at point $r$.

iii. Non-normal distribution of the error terms. Two distributions are used: a log-normal distribution and a Student-t distribution with 5, 10 or 15 degrees of freedom ($df$, in the following). The first allows us to measure the consequences of the asymmetry of the distribution function and the second provides information about the impact of outliers (a Student-t with few $df$ is prone to produce outliers).

iv. We will explore whether the existence of endogeneity in the data, omitted in the equations, affects the performance of the test. In order to do this, we simply introduce a linear relation between the error term and the regressor: iv.1) using a correlation coefficient of 0.2, low; iv.2) 0.59, medium; or, iv.3) 0.99, high.
v. We explore the behaviour of the Likelihood Ratio under several patterns of non-linearity using either: v.1) the sine function, \( y = \sin(y^*) \); v.2) the quadratic function, \( y = (y^*)^2 \); v.3) the inverse function, \( y = \frac{1}{y^*} \); v.4) the logarithm function of the absolute value, \( y = \log(|y^*|) \); v.5) a discretization of the data of a latent continuous variable, \( y^* \). In all cases, the \( y^* \) is obtained directly from expressions (12a) or (12b) of case of i). The discrete transformation of v.5 follows a single rule:

\[
y_r = \begin{cases} 
    0 & \text{if } y_r^* < y_{(k)}^* \\
    1 & \text{if } y_r^* \geq y_{(k)}^* 
\end{cases}
\]

(13)

where \( y_{(k)}^* \) stands for the k-th quantile of the latent variable \( \{y^*; r = 1, 2, \ldots; R\} \). We have used two values for the quantile, \( k = 0.7 \) and \( 0.5 \).

vi. As pointed out, among others, by Smith (2009) or Neuman and Mizruchi (2010), the use of dense weighing matrices has severe consequences on maximum likelihood estimation: the estimates are dramatically downward biased and most part of the ML tests loses power. We study this new case in a non-regular lattice support. Each experiment starts by obtaining a random set of spatial coordinates of each sample size (49, 100 or 225, respectively) in a two-dimensional space. Then we use the n nearest-neighbours criterion to build the corresponding weighting matrix. The values of \( n \) have been fixed as: \( n = [\alpha T]; \alpha = 0.05; 0.10; 0.25; 0.50 \) where \([\cdot]\) stands for the «integer part of».

### 3.2. Results of the Monte Carlo experiments

The Monte Carlo experiment provided us with a lot of results. In order to simplify, we focus on the frequency of rejection of the null hypothesis of the LR\(_{\text{COM}}\) test, at the 5% level of significance. Depending on the DGP used in the simulation, we estimate the size (a SEM model is in the DGP) or the power function of the test (we simulate a SLM model). It is well-known that the LR\(_{\text{COM}}\) is a good technique to discriminate between SEM and SLM models under ideal conditions. The interest now is to assess the behaviour of the test under non ideal circumstances.

Results are summarized in Figures 2 to 6. Figure 2 shows the performance of the LR\(_{\text{COM}}\) test under the three patterns of heteroscedasticity \((h1, h2 \text{ and } h3)\). The two non-normal distributions (the log-normal and the three cases for the Student-t) appear in Figure 3. Figure 4 shows the impact of the density of the weighting matrix on the LR\(_{\text{COM}}\) test whereas Figure 5 focuses on the case of endogeneity. Finally, in Figure 6 we evaluate the performance of the test for the five non-linear specifications. In all the Figures, «iid» corresponds to the control case (that is, ideal conditions).
Figure 2. Power and empirical size of LR\textsubscript{COM} test under heteroscedasticity
Figure 3. Power and empirical size of LR_{COM} test for non-normal distributions functions
Figure 4. Power and empirical size of $\text{LR}_{\text{COM}}$ test for dense weighting matrices
Figure 5. Power and empirical size of $LR_{COM}$ test under different degrees of endogeneity.
Figure 6. Power and empirical size of LR\textsubscript{COM} test under different patterns of non-linearity.
Figure 2 shows that heteroscedasticity affects negatively the performance of the test, especially in what respects to the power function. However, this is true only for the heteroscedasticity spatial patterns: the impact of the non-spatial heteroscedastic pattern ($h3$) is almost negligible, both on the power function and on the empirical size. The other two spatial heteroscedastic patterns ($h1$ and $h2$) suffer severe consequences slashing power and slightly raising the size.

The implications of the non-normality of the data are evident on Figure 4. The impact diminishes as the sample size increases. The asymmetry of the distribution function (log-normal case) seems to have a greater impact that the presence of outliers (Student-t case), especially for small sample sizes ($T = 49$ and $100$). The test tends to be slightly oversized in both cases. The situation is more balanced in large sample case where the size is correctly estimated.

The density of the weighting matrix has a clear impact in the behaviour of the $\text{LRCOM}$ test as it is clear in Figure 4 (the iid case corresponds to «5%», where each cell is connected with to the 5% of its neighbours). The use of dense matrices implies a tendency to slightly overestimate the size of the test, as it appears in the right panel, and severe losses in power especially for a range of intermediate values of the spatial dependence parameter. Denser matrices are a risk factor in spatial models that affects to almost every inference. The Common Factor test does not avoid these problems but the consequences are less severe than in other aspects.

Figure 5 shows that endogeneity has a very damaging effect on the $\text{LRCOM}$ test, especially in what respect to size (the iid case corresponds to «Corr.Coeff.0»). The figures of the right panel clearly indicate a strong tendency to reject, wrongly, the null of the $\text{LRCOM}$ test for intermediate to high values of the correlation coefficient between the regressor and the error term of the equation. Strong endogeneity means strong over-sizing. This tendency pushes upwards the power function estimated on the left panel (and obtained using the theoretical 5% significant value). Overall, these results indicate that endogeneity is a key issue in relation to the problem of model selection and that, at least for the $\text{LRCOM}$ test, a bootstrapping approach may be advisable.

Finally, the results for the non-linear processes offer a very heterogeneous picture as it is clear in Figure 6. First of all, looking at the right panels, there is a general tendency to underestimate the size even for very small values of the spatial dependence coefficient. In other words, we are going to select the SEM model more than the necessary. In relation to the power, we can identify three groups of functions: the quasi-linear functions (which includes the quadratic and the logarithm of the absolute value), the binary functions and the strongly non-linear functions (the sine and the inverse functions). The impact for the first group is small and the $\text{LRCOM}$ tends to work properly. For the case of discretized data, there is a noticeable power loss although the losses tend to diminish as the sample size increases. The left bottom panel indicates that 225 observations are not enough to guarantee a good power for intermediate to small values of the spatial dependence coefficient. In sum, it is clear that the presence of strong nonlinearities in the DGP is a challenge for this test that in some cases (i.e., the sine function) hardly detects SLM processes.
4. Conclusions

The tests of Common Factors were introduced into a spatial context at the beginning of the eighties when the current toolbox was still in its infancy. The Common Factor tests had never occupied a prominent role in this toolbox; only the Likelihood Ratio variant, the LRCOM, is popular. Habitually, these tests have been used in an auxiliary form, to corroborate conclusions obtained with other techniques. Nevertheless, we believe that the Common Factors Tests should play a more relevant role as a guide in applied work.

These tests should be used in connection with other techniques in order to explore the adequate direction for the specification process. At least, it should be borne in mind the requirement of Davidson (2000, p. 168): «The point is that although AR(1) errors may well be the correct specification, they impose a common-factor parameter restriction on the equation that requires to be tested. It would nowadays be regarded as bad practice to impose the AR(1) model without testing the implicit restriction».

Our position is that, given the peculiarities of the discipline, we must be a little more ambitious. Externalities and dynamic spatial relationships play a strategic role in any spatial model. These elements often have an evasive nature that makes them difficult to detect. For this reason, it is important to have techniques to discriminate between different spatial interaction mechanisms. The Common Factor tests may help in this problem.

The literature has paid attention to its performance under ideal conditions. For this reason, we tried to fill this gap by conducting a Monte Carlo experiment to evaluate its performance under some common non-ideal conditions: heteroscedasticity, non-normality, endogeneity, dense weighting matrices and non-linearity.

Our results have shown evidence on the following points. Regarding the empirical size of the test, results are quite acceptable except when there are endogenous regressors in the equation. As regards to the power, our results are very good in the case of endogeneity, and reasonably good also for the other cases. The worst situation corresponds to a spatial heteroscedastic pattern, to non-symmetric probability distribution functions and to strong departures of the assumption of linearity in the functional form (the sine function it is a pathological case).

In sum, we strongly suggest the use of the Likelihood Ratio test of Common Factors to spatial econometricians as a useful technique in the process of specifying a spatial model.

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